JOURNAL OF COMBINATORIAL THEORY, Series B 35, 319–322 (1983)

Note

On k-Graceful, Locally Finite Graphs

PETER J. SLATER

Mathematics Department, University of Alabama in Huntsville, Huntsville, Alabama 35899

Communicated by the Editors

Received May 3, 1982

While a finite, bipartite graph G can be k-graceful for every $k \geqslant 1$, a finite nonbipartite G can be k-graceful for only finitely many values of k. An improved bound on such possible values of k is presented. Definitions are extended to infinite graphs, and it is shown that if G is locally finite and vertex set $V(G)$ and edge set $E(G)$ are countably infinite, then for each $k \geqslant 1$ the graph G has a k-graceful numbering h mapping $V(G)$ *onto* the set of nonnegative integers.

1. INTRODUCTION

Rosa [11] introduced the study of numberings of graphs in a study of a problem of Ringel [10]. The study of what are called "k-graceful" graphs began with a problem in radio-astronomy. (See [2–5].) They are formally defined in [12, 13] and independently by Maheo and Thuillier [9]. Recently Hell [8] informed me that he too independently began studying k-gracefulness for an application to hashing functions.

Let N denote the set of positive integers and $N_0 = N \cup \{0\}$. For any graph G with vertex set $V(G)$ and edge set $E(G)$, and for any function $h: V(G) \to N_0$, the induced function $g_h: E(G) \to N_0$ is defined as follows. For each edge $x = (u, v)$ in $E(G)$, let $g_h(x) = g_h(u, v) = |h(u) - h(v)|$. Function h is called a *numbering* if h is one-to-one and g_h is also one-to-one. If G is finite, let $V(G) = \{v_1, v_2,..., v_n\}$ and $E(G) = \{x_1, x_2,..., x_e\}$. G is *k-graceful* if there exists a numbering $h: V(G) \to \{0, 1,..., k + e - 1\}$ for which $g_h(E(G)) = \{k, k + 1,..., k + e - 1\}$, and the numbering h making G k-graceful is called a *β_k-valuation*. An *α_k-valuation* of G is a β_k-valuation for which there is some L in $\{0, 1,..., k + e - 1\}$ such that for an arbitrary edge (u, v) in $E(G)$ either $h(u) \leqslant L < h(v)$ or $h(v) \leqslant L < h(u)$.

0095-8956/83 $3.00

THEOREM 1 [9, 13]. *If (bipartite) graph G has an α_j-valuation, then G is k-graceful for all $k \geqslant j$.*

In contrast, it is shown in [13] that if graph G is k-graceful and contains an odd cycle on $2j+1$ vertices, then $k \leqslant j(e-j-1)$. Strengthening this result, one can prove the next theorem.

THEOREM 2. *Let S be any subset of $E(G)$ and $P = \{E_1, E_2, ..., E_d\}$ a partition of S such that E_i is the edge set of an odd cycle C_i in G. Let n_m be the number of members of P of order m, and let $L = \sum_{j \geqslant 1} n_{2j+1}(j+1)$. If $k > (L-d)(e-L)/d - (d-1)/2$, then G is not k-graceful.*

Let $B(G) = \{k$: G *is* k-graceful$\}$. Similar to the conjecture I made for k-sequential graphs in [1] is the following.

Conjecture. For any set S of natural numbers there is a graph G_S with $B(G_S) = S$.

2. LOCALLY FINITE, COUNTABLY INFINITE GRAPHS

Assume G is a graph for which each of $V(G)$ and $E(G)$ is countably infinite, and suppose G is locally finite— that is, each vertex is incident with only finitely many edges. G is *k-graceful* if there exists a numbering h: $V(G) \to N_0$ for which $g_h : E(G) \to \{k, k+1, k+2, ...\}$ is a bijection, and such an h is a *β_k-valuation.*

Making use of an observation of Bloom [6] concerning the adjacency matrix of a gracefully labelled graph, Grace proved the following.

THEOREM 3 [7]. *If T is a countably infinite, locally finite tree, then T can be 1-gracefully labelled.*

One of the results in Slater and Vélez [14] implies the next result.

THEOREM 4 [14]. *If $P = v_1, v_2, v_3, ...$ is the infinite path, then P has a 1-graceful numbering h: $V(P) \to N_0$ with the additional property that h maps $V(P)$ onto N_0.*

Generalizing these results, graph G can be an arbitrary countably infinite graph—perhaps disconnected, perhaps with isolated vertices, etc.

THEOREM 5. *If G is a locally finite graph and $V(G)$ and $E(G)$ are countably infinite, then G has a k-graceful numbering h: $V(G) \to N_0$ which is a bijection.*

Proof. Select any listing of $V(G)$, say $L = (v_0, v_1, v_2, ...)$. For example, if

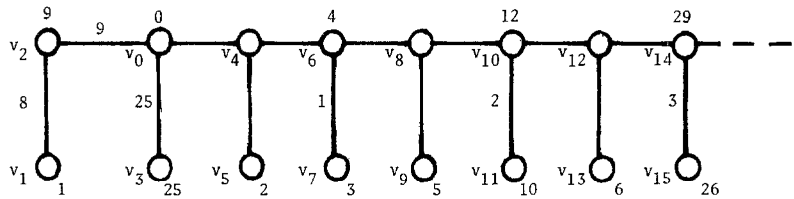

FIG. 1. Defining 1-graceful numbering h.

G is connected one can select v_0 arbitrarily, then list those vertices adjacent to it, then those at distance two from v_0, etc. The following procedure defines $h: V(G) \to N_0$ such that h is a bijection and $g_h: E(G) \to \{k, k+1, k+2, \ldots\}$ is a bijection.

Step 1 (Initialization). Let $h(v_0) = 0$, and let $M = 0$. (The value of M is kept equal to the largest current vertex label.)

Step 2. Repeat the following.

(A) Let i be the smallest value such that v_i is not yet labelled. Let $h(v_i) = 2M + k$, and replace M by $2M + k$. ($M \leftarrow 2M + k$.) Note that there is no duplication of vertex labels, nor can there be duplication of edge labels, and no edge receives a label less than k.

(B) Let i be the smallest value such that no vertex has yet received label i, and let j be the smallest value for which v_j and all vertices adjacent to v_j are, as yet, unlabelled. (Such a j exists because G is locally finite.) Let $h(v_j) = i$. Note that no edge receives a label. $M \leftarrow \max(M, i)$.

(C) Let i be the smallest value ($i \geqslant k$) so that no edge has yet received label i. Let j be the smallest value for which v_j is not isolated, and v_j, all vertices adjacent to v_j, and all vertices at distance two from v_j are, as yet, unlabelled. (Such a j exists because $E(G)$ is countably infinite and G is locally finite.) Let m be the minimum value with $(v_j, v_m) \in E(G)$. Let $h(v_j) = M + 1$ and $h(v_m) = M + 1 + i$. Note that the two new vertices receive distinct labels larger than any existing vertex label, and the only edge to receive a label is $x = (v_j, v_m)$ with $g_h(x) = i$. $M \leftarrow M + 1 + i$.

After Step 2 has been completed n times, $4n + 1$ vertices have been labelled including $v_0, v_1, \ldots, v_n$, vertex labels $0, 1, 2, \ldots, n$ have been used, and edge labels k, $k + 1, \ldots, k + n - 1$ have been used. Thus h will satisfy the requirements, and the proof is complete.

Figure 1 illustrates an infinite tree for which Step 2 has been performed three times.

REFERENCES

1. D. W. BANGE, A. E. BARKAUSKAS, AND P. J. SLATER, Simply sequential and graceful graphs, Proc. of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing, 1979, 155-162.
2. J. C. BERMOND, A. E. BROUWER, AND A. GERMA, Systèmes de triples et différences associées, *in* "Colloque CNRS, Problèmes Combinatoires et Theorie des Graphes, Orsay, 1976," pp. 35–38, C.N.R.S., Paris, 1978.
3. J. C. BERMOND, A. KOTZIG, AND J. TURGEON, On a combinatorial problem of antennas in radio-astronomy, *in* "Colloq. Math. Soc. János Bolyai 18, Combinatorics, Keszthély, Hungary, 1976," pp. 135–149, North-Holland, Amsterdam, 1978.
4. F. BIRAUD, E. J. BLUM, AND J. C. RIBES, On optimum synthetic linear arrays, *IEEE Trans. Antennas and Propagation* **AP22** (1974), 108–109.
5. F. BIRAUD, E. J. BLUM, AND J. C. RIBES, Some new possibilities of optimum synthetic linear arrays for radioastronomy, *Astronom. Astrophys.* **41** (1975), 409–413.
6. G. S. BLOOM, A chronology of the Ringel–Kotzig conjecture and the continuing quest to call all trees graceful, *in* "Topics in Graph Theory" (Scientist in Residence Program, N. Y.) (F. Harary, Ed.), pp. 32–51, Academy of Science, Washington, D.C., 1977.
7. T. GRACE, New proof techniques for gracefully labelling graphs, "Proc. of the Thirteenth S. E. Conference on Combinatorics, Graph Theory and Computing," 1982, pp. 403–408.
8. P. HELL, private communication.
9. M. MAHEO AND H. THUILLIER, On *d*-graceful graphs, *Ars Combin.* **13** (1982), 181–192.
10. G. RINGEL, *in* "Theory of Graphs and Its Applications," Proc. Symposium Smolenice, 1963 (Prague, 1964), 162.
11. A. ROSA, On certain valuations of the vertices of a graph, *in* "Theory of Graphs, Proc. Internat. Symposium Rome, 1966" (P. Rosenthiehl, Ed.), pp. 349–355, Dunod, Paris, 1968.
12. P. J. SLATER, On *k*-sequential and other numbered graphs, *Discrete Math.* **34** (1981), 185–193.
13. P. J. SLATER, On *k*-graceful graphs, Proc. of the Thirteenth S. E. Conference on Combinatorics, Graph Theory and Computing, 1982, pp. 53–57.
14. P. J. SLATER AND W. Y. VÉLÉZ, Permutations of the positive integers with restrictions on the sequence of differences, *Pacific J. Math.* **71** (1977), 193–196.